

# Noncommutative Yang-Mills and noncommutative relativity: a bridge over troubled water

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**Abstract.** Connes' view at Yang-Mills theories is reviewed with special emphasis on the gauge invariant scalar product. This landscape is shown to contain Chamseddine and Connes' noncommutative extension of general relativity restricted to flat spacetime, if the top mass is between 172 and 204 GeV. Then the Higgs mass is between 188 and 201 GeV.

## 1 Introduction

Einstein was a passionate sailor. We speculate that this was no accident. The subtle harmony between geometries and forces becomes palpable to the sailor, he sees the curvature of the sail and feels the force that it produces. Before Einstein, it was generally admitted that forces are vector fields in an Euclidean space,  $\mathbb{R}^3$ , the scalar product being necessary to define work and energy. Einstein generalized Euclidean to Minkowskian and Riemannian geometry and we have two *dreisätze* or *règles de trois*. Take Coulomb's static law for the electric field with coupling constant  $\epsilon_0$  and add Minkowskian geometry with its scale  $c$ , the speed of light: you obtain Maxwell's theory. In particular, there appears the magnetic field with feeble coupling constant  $\mu_0 = 1/(c^2\epsilon_0)$ . Maxwell's theory is celebrated today as Abelian or should we say, commutative Yang-Mills theory. The second *dreisatz* starts from Newton's (static) universal law of gravitation, adds Riemannian geometry to obtain general relativity with new feeble, gravito-magnetic forces.

Connes proposes two more *dreisätze*. Take a certain Yang-Mills theory with coupling constant  $g$ , coupled to a Dirac spinor of mass  $m$ . Add noncommutative geometry [1] with an energy scale  $\Lambda$ : you obtain a Yang-Mills-Higgs theory [2,3]. The symmetry breaking scalar becomes a magnetic field of the Yang-Mills field and its mass and self-coupling  $\Lambda$  are constrained in terms of  $g$ ,  $m$  and  $\Lambda$ . His second *dreisatz* starts from general relativity, adds noncommutative geometry to obtain the Einstein-Hilbert action plus the Yang-Mills-Higgs action [4,5]. Now the Yang-Mills and the Higgs fields are magnetic fields of the gravitational field. Again there are constraints on  $\lambda$ , but they are different.

Let us call noncommutative Yang-Mills the third and noncommutative relativity the fourth *dreisatz*. Note how-

ever that – unlike with supersymmetry – you cannot take any Yang-Mills theory and put 'noncommutative' in front [6–8]. Note also that, behind noncommutative relativity, there stands a genuine noncommutative extension of Einstein's principle of general relativity, the spectral principle.

One of the attractive features of noncommutative geometry is to unify gauge couplings with scalar self-couplings and Yukawa couplings. These couplings will be at the center of our discussion. They are related to the set of all scalar products on a given space, which is a cone: in the case of gauge invariant scalar products on the Lie algebra of a Yang-Mills theory, the gauge couplings are positive coordinates on this cone. For us, a noncommutative geometry is given by a spectral triple  $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ . The motivation for this noncommutative geometry comes from Connes' theorem [4] establishing a one-to-one correspondence between commutative spectral triples and Riemannian spin geometries [9].  $\mathcal{A}$  is an associative involution algebra with unit, represented faithfully on a Hilbert space  $\mathcal{H}$  on which also the self-adjoint 'Dirac' operator  $\mathcal{D}$  acts. Physically, the Hilbert space is spanned by the fermions. The Dirac operator serves several purposes: it defines the kinetic term for fermions, it allows to construct differential forms – the Yang-Mills fields are 1-forms –, it is used as ultra-violet regulator to define the scalar product and above all, it defines the metric structure of space-time.

So, in this paper, playing with the scalar products and the experimental accuracies of the gauge couplings and the masses, we consider the noncommutative Yang-Mills and noncommutative relativity theories as effective theories, using the renormalization flow and show that they can be related according to certain constraints on the Higgs and top masses.

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## 2 Noncommutative relativity

The dynamical variable of gravity is the metric on space-time. Einstein used the matrix  $g^{\mu\nu}(x)$  of the metric  $g$  with respect to a coordinate system  $x^\mu$  to parameterize the set of all metrics on a fixed space-time  $M$ . The coordinate system being unphysical, Einstein required his field equations for the metric to be covariant under coordinate transformations, the principle of general relativity. Elie Cartan used tetrads, *repères mobiles*, to parameterize the set of all metrics. This parameterization allows to generalize the Dirac operator  $\mathcal{D}$  to curved space-times and also reformulates general relativity as a gauge theory under the Lorentz group. Connes [4] goes one step further by relating the set of all metrics to the set of all Dirac operators. The Einstein-Hilbert action, from this point of view, is the Wodzicki residue of the second inverse power of the Dirac operator [10] and is computed most conveniently from the second coefficient of the heat kernel expansion of the Dirac operator squared. The heat kernel expansion [11] is an old friend [12] from quantum field theory in curved space-time, from its formal relation to the one-loop effective action

$$S_{\text{eff}} = \text{tr} \log(\mathcal{D}^2/\Lambda^2), \quad \Lambda \text{ a cutoff.} \quad (1)$$

This relation has been used by Sakharov [13] to induce gravity from quantum fluctuations, leading however to a negative Newton constant [14].

By generalizing the metric, the Dirac operator plays a fundamental role in noncommutative geometry. To describe Yang-Mills theories, Connes considers the tensor product of space-time and internal space in this new geometry, a natural point of view because the fermionic mass matrix qualifies as Dirac operator on internal space. This cheap tensor product unifies spacetime diffeomorphisms with internal gauge transformations by extending Einstein's principle of general relativity to noncommutative geometry. Remember that Einstein constructed general relativity in two steps, by applying his principle first to matter, then to the gravitational field itself. Connes follows this pattern and of course in his case, the spinors are the matter.

To generalize the Dirac operator from flat to curved space-time (locally), it is sufficient to write the Dirac operator first in flat space-time but with respect to noninertial coordinates. A straightforward calculation produces the *covariant* Dirac operator that contains the spin connection  $\omega$ . Although of vanishing curvature,  $\omega$  contains already a lot of physics, e.g. the centrifugal and Coriolis accelerations in the coordinates of the rotating disk, the quantum interference pattern of neutrons [15] in oscillating coordinates. Then, the generalization to curved space is easy where  $\omega$  describes the (minimal) coupling of the spinor to the gravitational field. In Einstein's spirit, the covariant Dirac operator is obtained by acting with the diffeomorphism group on the flat Dirac operator. But the diffeomorphism group is just the automorphism group of the associative (and commutative) algebra  $\mathcal{C}^\infty(M)$  representing space-time in noncommutative geometry. On the other

hand, the product of space-time and internal space is represented in this geometry by the tensor product of  $\mathcal{C}^\infty(M)$  with a matrix algebra. Its automorphism group is the semi-direct product of the diffeomorphisms and unitaries of the matrix algebra, the internal gauge transformations. The diffeomorphisms are the outer, the gauge transformations are the inner automorphisms. And what do we get when this entire automorphism group acts on the flat Dirac operator? We get the total covariant Dirac operator containing the spin connection, the gauge connection and the Higgs [3]. In other words, we get the minimal couplings of the Dirac spinor to the gravitational and Yang-Mills fields and its Yukawa couplings to the Higgs field. In Connes' words, the Higgs and Yang-Mills fields are noncommutative fluctuations of the metric. (Abelian Yang-Mills theories do not have such fluctuations.) Accordingly, Connes generalizes Einstein's principle of general relativity by postulating that only the intrinsic properties of the covariant Dirac operator be relevant for physics. Here intrinsic means invariant under automorphisms. Thus, these properties must concern the spectrum only. Spectral principle is a convenient name for Connes' generalization of the principle of general relativity.

So far, we have only the kinematics of the metric (and its fluctuations). To get its dynamics, Einstein developed the full power of the principle of general relativity and derived the Einstein-Hilbert action. In short this story: the  $1/r^2$  in Newton's universal law is the Green function of the divergence, an operator of first order in the forces. We already know that the forces are encoded in the connection  $\omega$ . Riemannian geometry tells us that the connection is obtained from first order derivatives of the metric. Therefore Einstein looked for a second order differential equation for the metric. The covariance under change of coordinates fixes this equation up to the cosmological constant to be the Einstein equation. Chamseddine & Connes [5] reedit this story using the spectral principle. It is stronger than Einstein's principle in the sense that for the metric only, the Einstein-Hilbert action follows without the use of Newton's law. In addition, the spectral principle fixes the action of the fluctuations to be the Yang-Mills action, the covariant Klein-Gordon action and the symmetry breaking Higgs potential. Warning: following physicists' habits, we have confused diffeomorphisms and coordinate transformations. Cleaning up this point leads to deep mathematics [16] and probably a further unification of general relativity and Yang-Mills theory: the reduction of the diffeomorphism group to an isometry group might take the form of a spontaneous symmetry break down.

### 2.1 The stiff action

In even dimensions, the spectrum of the Dirac operator is even and it is sufficient to consider the positive part of the spectrum which in the Euclidean is conveniently characterized by a distribution function

$$S = \text{tr} f(\mathcal{D}^2/\Lambda^2), \quad (2)$$

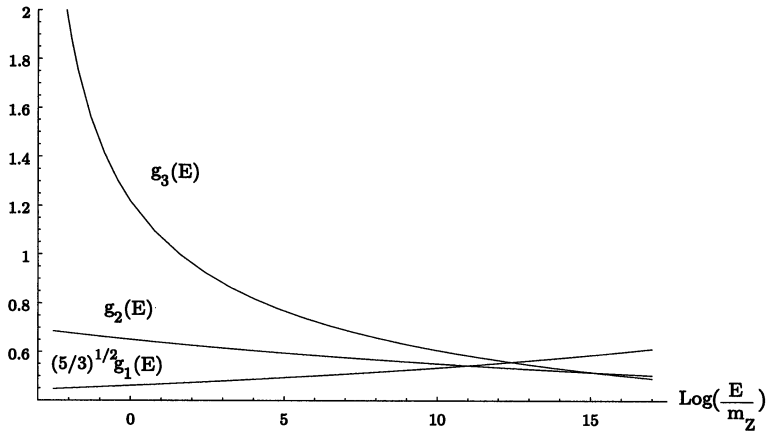


Fig. 1. The evolution of the three coupling constants

where  $\Lambda$  is an energy cutoff and  $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is a positive, smooth function with finite, strictly positive first ‘momenta’,

$$f_0 := \int_0^\infty u f(u) du, \quad f_2 := \int_0^\infty f(u) du, \quad f_4 := f(0). \tag{3}$$

If instead,  $f$  was the logarithm, this trace, after a proper renormalization, would be Sakharov’s induced gravity action. The positive function  $f$  is universal: the action  $S$  can be computed asymptotically [17], that is up to terms of the order of  $\Lambda^{-2}$ , using the Lichnérowicz formula and the heat kernel expansion. The action depends only on the three momenta  $f_0, f_2, f_4$  and takes the form:

$$\begin{aligned} \text{tr } f(\mathcal{D}_{t,\text{cov}}^2/\Lambda^2) \approx & \int_M \left[ -\frac{1}{16\pi} m_P(\Lambda)^2 R + \Lambda_C(\Lambda) \right. \\ & + \frac{1}{2} g_3(\Lambda)^{-2} \text{tr } F_{\mu\nu}^{(3)*} F^{(3)\mu\nu} + \frac{1}{2} g_2(\Lambda)^{-2} \text{tr } F_{\mu\nu}^{(2)*} F^{(2)\mu\nu} \\ & + \frac{1}{4} g_1(\Lambda)^{-2} F_{\mu\nu}^{(1)*} F^{(1)\mu\nu} + \frac{1}{2} (D_\mu \varphi)^* D^\mu \varphi \\ & + \lambda(\Lambda) |\varphi|^4 - \frac{1}{2} \mu(\Lambda)^2 |\varphi|^2 - a(\Lambda) C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} \\ & \left. + \frac{1}{12} |\varphi|^2 R \right] (\det g_{..})^{1/2} d^4x. \end{aligned} \tag{4}$$

Here  $\mathcal{D}_{t,\text{cov}}$  is the total, covariant Dirac operator of the standard model of electroweak and strong interactions with  $N = 3$  generations of quarks and leptons. It follows that  $\varphi$  is an isospin doublet. After a proper normalization of the kinetic terms and a shift of the Higgs field by its vacuum expectation value,  $|\varphi| = v(\Lambda) = \mu(\Lambda)/(2\sqrt{\lambda(\Lambda)})$ , we can identify Newton’s constant  $G = \hbar c m_P^{-2}$ , the cosmological constant  $\Lambda_C$  and the other coupling constants

$$m_P(\Lambda)^2 = \frac{1}{3\pi} f_2 \left[ 15N - 2 \frac{L^2}{Q} \right] \Lambda^2, \tag{5}$$

$$L(\Lambda) := 3(m_t^2 + m_c^2 + m_u^2 + m_b^2 + m_s^2 + m_d^2) + m_\tau^2 + m_\mu^2 + m_e^2, \tag{6}$$

$$Q(\Lambda) := 3(m_t^4 + m_c^4 + m_u^4 + m_b^4 + m_s^4 + m_d^4) + m_\tau^4 + m_\mu^4 + m_e^4, \tag{7}$$

$$\Lambda_C(\Lambda) = \frac{1}{4\pi^2} \left[ 15N f_0 - \frac{f_2^2 L^2}{f_4 Q} \right] \Lambda^4, \tag{8}$$

$$g_3(\Lambda)^{-2} = \frac{N}{3\pi^2} f_4, \tag{9}$$

$$g_2(\Lambda)^{-2} = \frac{N}{3\pi^2} f_4, \tag{10}$$

$$g_1(\Lambda)^{-2} = \frac{5}{3} \frac{N}{3\pi^2} f_4, \tag{11}$$

$$\lambda(\Lambda)^{-1} = \frac{1}{\pi^2} f_4 \frac{L(\Lambda)^2}{Q(\Lambda)} \tag{12}$$

$$= \frac{3}{\pi^2} f_4 \left( 1 + 2 \frac{m_b(\Lambda)^2}{m_t(\Lambda)^2} + O\left(\frac{m_\tau(\Lambda)^2}{m_t(\Lambda)^2}\right) \right),$$

$$\mu(\Lambda)^2 = 2 \frac{f_2}{f_4} \Lambda^2, \tag{13}$$

$$a(\Lambda) = \frac{3N}{64\pi^2} f_4. \tag{14}$$

From now on, we ignore the gravitational part because we want to use the renormalization flow of the coupling constants and also because we want to compare this theory with the noncommutative Yang-Mills that, by the way, automatically has a vanishing cosmological constant as we shall see.

The constraints for the gauge couplings,

$$g_3(\Lambda) = g_2(\Lambda) \text{ and } \sin^2 \theta_w(\Lambda) = \frac{g_2^{-2}}{g_1^{-2} + g_2^{-2}} = \frac{3}{8}$$

(9–11), are familiar from grand unification and force us to assume the big desert. Consequently all numerical considerations will be qualitative only. Indeed, the three gauge couplings  $g_i(\Lambda)$ , once fixed at the  $Z$ -mass to their experimental values  $\sin^2 \theta_w(m_Z) = 0.2315 \pm 0.0005$ , see appendix, do not meet in a point anymore as was the case in the  $SU(5)$  days. Today they define a triangle with  $\Lambda = 10^{13} - 10^{17}$  GeV and  $\sqrt{\frac{5}{3}} g_1(\Lambda), g_2(\Lambda), g_3(\Lambda)$  are in the interval 0.52–0.56, Fig. 1. Details on the renormalization group flow can be found in the appendix. For the noncommutative constraints (9–11), this means that  $f_4$  cannot take a precise value,  $\frac{1}{4\pi^2} f_4 = 0.80 - 0.94$ .

## 2.2 The soft Einstein-Hilbert action

Of course, we may try to do better by introducing more parameters. Let  $z'$ , the ‘noncommutative coupling constant’, be a positive operator on the fermionic Hilbert space that commutes with the representation and the Dirac operator. For the standard model, this  $z'$  contains four positive numbers  $x', y'_1, y'_2, y'_N$ . We soften the action (4) to  $\text{tr}[z' f(\mathcal{D}_{t,\text{cov}}^2/\Lambda^2)]$ . Then the constraints read [18]:

$$g_3(\Lambda)^{-2} = \frac{1}{9\pi^2} f_4 N x', \quad (15)$$

$$g_2(\Lambda)^{-2} = \frac{1}{12\pi^2} f_4 (N x' + y'_1 + y'_2 + y'_N), \quad (16)$$

$$g_1(\Lambda)^{-2} = \frac{1}{12\pi^2} f_4 \left( \frac{11}{9} N x' + 3(y'_1 + y'_2 + y'_N) \right), \quad (17)$$

$$\lambda(\Lambda)^{-1} = \frac{1}{\pi^2} f_4 \frac{L(\Lambda)^2}{Q(\Lambda)} \quad (18)$$

$$L(\Lambda) = x'(m_t^2 + m_c^2 + m_u^2 + m_b^2 + m_s^2 + m_d^2) + y'_3 m_\tau^2 + y'_2 m_\mu^2 + y'_1 m_e^2, \quad (19)$$

$$Q(\Lambda) = x'(m_t^4 + m_c^4 + m_u^4 + m_b^4 + m_s^4 + m_d^4) + y'_3 m_\tau^4 + y'_2 m_\mu^4 + y'_1 m_e^4, \quad (20)$$

$$\mu(\Lambda)^2 = 2 \frac{f_2}{f_4} \Lambda^2 \quad (21)$$

If  $z' = 1_{90}$ , then  $x' = 3, y'_1 = y'_2 = y'_N = 1$  and we recover the stiff relations (11–13).

## 2.3 The dominating top approximation and renormalization flow

The soft relations do not have the problem  $g_3(\Lambda) = g_2(\Lambda)$  anymore, but we still cannot avoid the desert. In fact now

$$\sin^2 \theta_w(\Lambda) = \frac{N x' + (y'_1 + y'_2 + y'_N)}{\frac{20}{9} N x' + 4(y'_1 + y'_2 + y'_N)},$$

and the weak mixing angle is constrained for all  $\Lambda$ :  $\frac{1}{4} < \sin^2 \theta_w(\Lambda) < \frac{9}{20}$ . From now on, we neglect all fermion masses with respect to the top mass. This approximation induces relative errors of the order of  $m_b^2/m_t^2 = 0.0006$  and it reduces the number of positive parameters from four,  $x', y'_1, y'_2, y'_N$  to two,  $x'$  and  $y' := y'_1 + y'_2 + y'_N$ . In the one loop approximation, the evolution of the gauge couplings (49–51) decouples from the other couplings and we can solve the constraints (15–17) such that at the  $Z$  mass, they reproduce precisely the experimental values. The last non-empty constraint (18),  $\lambda(\Lambda) = \frac{N}{9} g_3(\Lambda)^2$  then fixes the Higgs mass. With these approximations, we obtain:

– In the stiff case,  $x' = y' = 3$ , the uncertainty on the cutoff is large:

$$\left. \begin{aligned} \Lambda &= (10^{13} - 10^{17}) \text{ GeV}, \\ \frac{1}{12\pi^2} f_4 x' &= \frac{1}{12\pi^2} f_4 y' = 0.80 - 0.94, \\ m_H &= 182 \pm 10 \pm 2 \pm 7 \text{ GeV}. \end{aligned} \right\} \text{stiff EH}$$

The first error is from the uncertainty in the noncommutative scale  $\Lambda$ , the second from the present experimental uncertainty in the gauge couplings,  $g_3 = 1.218 \pm 0.026$ , and the third from the uncertainty in the top mass,  $m_t = 175 \pm 6 \text{ GeV}$ .

– In the soft case, the cutoff is sharp:

$$\left. \begin{aligned} \Lambda &= 0.96 \cdot 10^{10} \text{ GeV}, \\ \frac{1}{12\pi^2} f_4 x' &= 0.578, \\ \frac{1}{12\pi^2} f_4 y' &= 1.369, \\ m_H &= 190 \pm 0 \pm 1 \pm 4 \text{ GeV}. \end{aligned} \right\} \text{soft EH}$$

Let us anticipate that this comparison will work quantitatively only for the stiff case. So if we spell out the soft case here, then not because we believe that it makes sense to fit numbers through the big desert with the indicated precision. Our aim is to assess the stability of the Higgs mass prediction and also to make the comparison with the noncommutative Yang-Mills easier.

## 3 The noncommutative Yang-Mills action

After this quick review of the noncommutative version of general relativity in flat space-time, we now turn to our main concern, noncommutative Yang-Mills theory, Connes’ first dreisatz. The point will be that Connes’ second dreisatz adds insight to the older one.

Noncommutative geometry does to space-time  $M$ , a Riemannian manifold, what quantum mechanics did to phase space. An uncertainty relation is introduced by allowing the commutative algebra of functions  $C^\infty(M)$  to become noncommutative. Let us call  $\mathcal{A}$  this new algebra that we still suppose real, associative and equipped with a unit and an involution. On phase space,  $\mathcal{A}$  was just the algebra of observables. Now we want to define a distance on this new space that has lost its points. Following Connes, we need a faithful representation  $\rho$  of  $\mathcal{A}$  via bounded operators on a Hilbert space  $\mathcal{H}$ , the space of fermions, and a selfadjoint ‘Dirac’ operator  $\mathcal{D}$  on  $\mathcal{H}$ . Connes calls these three ingredients a spectral triple,  $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ . They satisfy axioms that are simply taken from the properties of the commutative case,  $\mathcal{A} = C^\infty(M)$ , the Hilbert space  $\mathcal{H}$  is the space of ordinary, square integrable Dirac spinors. An element  $f$  of  $\mathcal{A}$  is a differentiable function on space-time,  $f(x)$ , and it acts on a spinor  $\psi(x)$  by multiplication  $(\rho(f)\psi)(x) := f(x)\psi(x)$ .  $\mathcal{D} = \not{\partial}$  is the ordinary Dirac operator. Only recently Connes has completed the list of axioms [4] as to have a one-to-one correspondence between commutative spectral triples and Riemannian spin manifolds. To this end, he needed two other old friends from particle physics, a chirality operator  $\chi$  and a real structure  $J$ . The chirality is a unitary operator of square one that commutes with the representation. Therefore  $\chi$  decomposes the representation space into a left-handed piece  $\frac{1-\chi}{2}\mathcal{H}$  and a right-handed piece  $\frac{1+\chi}{2}\mathcal{H}$ . In the commutative case, of course  $\chi = \gamma_5$ . The real structure is an anti-unitary operator that in the commutative case reduces to the charge conjugation operator  $C$ .  $J$  is of square plus or

minus one, depending on space-time dimension and signature. The dimensionality of  $M$  can be recovered from the spectrum of the Dirac operator. Indeed for compact manifolds, the spectrum is discrete and the ordered eigenvalues  $\lambda_n$  grow like  $n^{1/\dim M}$ . This motivates the name spectral triple. Let us mention one more axiom. The Poincaré duality on manifolds is promoted to an axiom in quite an abstract form. We anticipate that, in the case of the standard model, this Poincaré duality will prohibit right-handed neutrinos [3].

### 3.1 Scalar products

To construct a Yang-Mills action  $\int \text{tr} F * F$ , we need four ingredients: differential forms on spacetime  $M$ , a Lie group  $G$ , ‘the internal space’, a scalar product on the space of differential forms  $\Omega M$  and an invariant scalar product on the Lie algebra  $\mathfrak{g}$  of the group  $G$ . The gauge field  $A$  is a 1-form with values in  $\mathfrak{g}$ , its field strength or curvature is the 2-form  $F := dA + \frac{1}{2}[A, A]$  again with values in  $\mathfrak{g}$ . To construct the action which is a real number, we take the scalar products of the field strength with itself. The first scalar product involves the space-time metric  $g$  hidden in the Hodge star  $*$ ,  $\langle \omega, \kappa \rangle = \int_M \omega^* * \kappa$ ,  $\omega$  and  $\kappa$  differential forms of same degree. In components, e.g. for 2-forms  $\omega = \frac{1}{2} \omega_{\mu\nu} dx_\mu dx_\nu$ , we have  $\langle \omega, \kappa \rangle = \frac{1}{2} \int_M \omega_{\mu\nu}^* \kappa_{\mu'\nu'} g^{\mu\mu'} g^{\nu\nu'} (\det g)^{1/2} d^4x$ . We suppose  $M$  Euclidean, otherwise this scalar product would only be a pseudo scalar product. The second scalar product is on the Lie algebra. It only exists if the Lie group is compact. E.g. for  $G = SU(n)$ , the general invariant scalar product is  $\langle a, b \rangle = \frac{2}{g_n} \text{tr}(a^* b)$ ,  $a, b \in \mathfrak{su}(n)$  and the coupling constant  $g_n$  is a positive number. In general, the space of all scalar products is a cone whose coordinates are the coupling constants.

To play the noncommutative Yang-Mills game, we need a scalar product for differential forms. For our notations and the construction of noncommutative differential forms the reader is referred to [19]. In the noncommutative context, the scalar product has another utility. It allows us to interpret the noncommutative differential forms in  $\Omega_{\mathcal{D}} \mathcal{A}$  not as classes but as concrete operators on the Hilbert space  $\mathcal{H}$ : degree by degree, we embed  $\Omega_{\mathcal{D}}^p$  in  $\pi(\Omega^p \mathcal{A})$  as orthogonal complement of  $\mathcal{J}^p$ . If  $\mathcal{H}$  was finite dimensional, we would naturally take as scalar product of two operators  $\omega$  and  $\kappa$ ,  $\langle \omega, \kappa \rangle = \text{tr}(\omega^* \kappa)$ . For infinite dimensional Hilbert spaces, we have to regularize and we use the Dirac operator to do so. Thanks to the asymptotic behavior of its spectrum,  $\text{tr}[\omega^* \kappa |\mathcal{D}|^{-\dim}]$  only diverges logarithmically. The Dixmier trace  $\text{tr}_{\text{Dix}}$  gets rid of this divergence [20] and we have a natural scalar product:

$$\langle \omega, \kappa \rangle = \text{Re tr}_{\text{Dix}}[\omega^* \kappa |\mathcal{D}|^{-\dim}], \quad \omega, \kappa \in \pi(\Omega^p \mathcal{A}).$$

We denote by  $(\cdot, \cdot)$  its restriction to  $\Omega_{\mathcal{D}} \mathcal{A}$ . In the commutative case of a four dimensional manifold  $M$ , these scalar products are independent of  $M$ :

$$\langle \omega, \kappa \rangle = \frac{1}{32\pi^2} \text{Re} \int_M \text{tr}_4[\omega^* \kappa] d^4x, \quad \omega, \kappa \in \pi(\Omega^p \mathcal{A}),$$

$$\langle \omega, \kappa \rangle = \frac{1}{8\pi^2} \text{Re} \int_M \omega^* * \kappa, \quad \omega, \kappa \in \Omega_{\mathcal{D}}^p \mathcal{A},$$

where we view the quotient by the junk as subspace orthogonal to the junk. We anticipate that this scalar product will also induce the one we need on the Lie algebra. In order to get the coupling constants, we soften the scalar products to:

$$\langle \omega, \kappa \rangle_z = \text{Re tr}_{\text{Dix}}[z \omega^* \kappa |\mathcal{D}|^{-\dim}], \quad \omega, \kappa \in \tau(\Omega^p \mathcal{A}) \quad (22)$$

$$\langle \omega, \kappa \rangle_z = \text{Re tr}_{\text{Dix}}[z \omega^* \kappa |\mathcal{D}|^{-\dim}], \quad \omega, \kappa \in \Omega_{\mathcal{D}}^p \mathcal{A}. \quad (23)$$

$z$  is a positive operator on Hilbert space that commutes with  $\rho$ ,  $J\rho J^{-1}$ ,  $\mathcal{D}$  and  $\chi$ . Whether or not  $z$  commutes with  $J$  will be a difficult choice. In the commutative case, we have anyhow that  $z$  is proportional to the identity.

### 3.2 The standard model

#### 3.2.1 The algebraic setting

The standard model has an internal space that does fit the elaborate axioms of a spectral triple. The internal spectral triple of the standard model is not far from being the simplest, non-degenerate example. To make this more precise, we note that the standard model viewed as an ordinary Yang-Mills-Higgs theory has the following four *unrelated* features:

- (i) weak interactions break parity maximally,
- (ii) weak interactions suffer spontaneous break down,
- (iii) strong interactions do not break parity,
- (iv) strong interactions do not suffer spontaneous break down.

Flip just one of these features and the standard model is outside the noncommutative axioms [7,21]. A more quantitative constraint concerns the Higgs representation, that in Connes’ formulation is not chosen but computed. The spectral triple of the standard model implies that the Higgs scalar transforms as one doublet under weak isospin entailing a unit  $\rho$ -factor,

$$\rho := \frac{m_W^2}{\cos^1(\theta_w) m_Z^2} = 1.$$

Experimentally we have today  $\rho = 1.0012 \pm 0.0031$ .

The geometric version of the standard model is well documented in the literature [1–3,22] and we just have to fix our notations. We denote by  $\mathbb{H}$  the algebra of quaternions, viewed as  $2 \times 2$  matrices,

$$\begin{pmatrix} x & -\bar{y} \\ y & \bar{x} \end{pmatrix} \in \mathbb{H}, \quad x, y \in \mathbb{C}.$$

We concentrate on the internal spectral triple. All its ingredients are finite dimensional,

$$\begin{aligned} \mathcal{A} &= \mathbb{H} \oplus \mathbb{C} \oplus M_3(\mathbb{C}) \ni (a, b, c), \\ \mathcal{H}_L &= (\mathbb{C}^2 \otimes \mathbb{C}^N \otimes \mathbb{C}^3) \oplus (\mathbb{C}^2 \otimes \mathbb{C}^N \otimes \mathbb{C}), \\ \mathcal{H}_R &= ((\mathbb{C} \oplus \mathbb{C}) \otimes \mathbb{C}^N \otimes \mathbb{C}^3) \oplus (\mathbb{C} \otimes \mathbb{C}^N \otimes \mathbb{C}). \end{aligned}$$

In each summand of the Hilbert spaces, the first factor denotes weak isospin doublets or singlets, the second  $N$  generations,  $N = 3$ , and the third denotes color triplets or singlets. Let us choose the following basis of  $\mathcal{H} = \mathbb{C}^{90}$ :

$$\begin{aligned} & \begin{pmatrix} u \\ d \end{pmatrix}_L, \begin{pmatrix} c \\ s \end{pmatrix}_L, \begin{pmatrix} t \\ b \end{pmatrix}_L, \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L, \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}_L, \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix}_L; \\ & u_R, c_R, t_R, e_R, \mu_R, \tau_R; \\ & d_R, s_R, b_R, \\ & \begin{pmatrix} u \\ d \end{pmatrix}_L^c, \begin{pmatrix} c \\ s \end{pmatrix}_L^c, \begin{pmatrix} t \\ b \end{pmatrix}_L^c, \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L^c, \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}_L^c, \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix}_L^c; \\ & u_R^c, c_R^c, t_R^c, e_R^c, \mu_R^c, \tau_R^c. \end{aligned}$$

The representation  $\rho$  acts on  $\mathcal{H}$  by

$$\begin{aligned} \rho(a, b, c) &:= \begin{pmatrix} \rho_w(a, b) & 0 \\ 0 & \rho_s(b, c) \end{pmatrix} \\ &:= \begin{pmatrix} \rho_{wL}(a) & 0 & 0 & 0 \\ 0 & \rho_{wR}(b) & 0 & 0 \\ 0 & 0 & \rho_{sL}(b, c) & 0 \\ 0 & 0 & 0 & \rho_{sR}(b, c) \end{pmatrix} \end{aligned}$$

with

$$\begin{aligned} \rho_{wL}(a) &:= \begin{pmatrix} a \otimes 1_N \otimes 1_3 & 0 \\ 0 & a \otimes 1_N \end{pmatrix}, \\ \rho_{wR}(b) &:= \begin{pmatrix} B \otimes 1_N \otimes 1_3 & 0 \\ 0 & \bar{b} 1_N \end{pmatrix}, \quad B := \begin{pmatrix} b & 0 \\ 0 & \bar{b} \end{pmatrix}, \\ \rho_{sL}(b, c) &:= \begin{pmatrix} 1_2 \otimes 1_N \otimes c & 0 \\ 0 & \bar{b} 1_2 \otimes 1_N \end{pmatrix}, \\ \rho_{sR}(b, c) &:= \begin{pmatrix} 1_2 \otimes 1_N \otimes c & 0 \\ 0 & \bar{b} 1_N \end{pmatrix}. \end{aligned}$$

The chosen representation  $\rho$  will take into account weak interactions  $\rho_w(a, b)$ ,  $a \in \mathbb{H}$ ,  $b \in \mathbb{C}$ , and strong interactions  $\rho_s(b, c)$ ,  $c \in M_3(\mathbb{C})$ ,  $c$  for color. This choice discriminates between leptons (color singlets) and quarks (color triplets). The role of  $b \in \mathbb{C}$  appearing in both weak interactions  $\rho_w(a, b)$  and strong interactions  $\rho_s(b, c)$  is crucial to make  $\rho(a, b, c)$  a representation of  $\mathcal{A}$  and is crucial for weak hypercharge computations. There is an apparent asymmetry between particles and anti-particles, the former are subject to weak, the latter to strong interactions. However, since particles and anti-particles are permuted in the covariant Dirac operator by

$$J = \begin{pmatrix} 0 & 1_{15N} \\ 1_{15N} & 0 \end{pmatrix} \circ c.c.,$$

the theory is invariant under charge conjugation. We denote the complex conjugation by  $c.c.$ . For completeness, we record the chirality as matrix

$$\chi = \begin{pmatrix} -1_{8N} & 0 & 0 & 0 \\ 0 & 1_{7N} & 0 & 0 \\ 0 & 0 & -1_{8N} & 0 \\ 0 & 0 & 0 & 1_{7N} \end{pmatrix}.$$

The third item in the spectral triple is the Dirac operator

$$\mathcal{D} = \begin{pmatrix} 0 & \mathcal{M} & 0 & 0 \\ \mathcal{M}^* & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The fermionic mass matrix of the standard model is

$$\mathcal{M} = \begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes M_u \otimes 1_3 + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes M_d \otimes 1_3 & 0 \\ 0 & \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes M_e \end{pmatrix},$$

with

$$\begin{aligned} M_u &:= \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_c & 0 \\ 0 & 0 & m_t \end{pmatrix}, \\ M_d &:= C_{KM} \begin{pmatrix} m_d & 0 & 0 \\ 0 & m_s & 0 \\ 0 & 0 & m_b \end{pmatrix}, \\ M_e &:= \begin{pmatrix} m_e & 0 & 0 \\ 0 & m_\mu & 0 \\ 0 & 0 & m_\tau \end{pmatrix}. \end{aligned}$$

All indicated fermion masses are supposed positive and different. The Cabibbo-Kobayashi-Maskawa matrix  $C_{KM}$  is supposed non-degenerate in the sense that there is no simultaneous mass and weak interaction eigenstate. Note that the strong interactions are vector-like and  $\rho_s$  commutes with  $\mathcal{D}$ .

Let us compute the noncommutative coupling constant  $z$ . We recall that  $z$  is a positive operator on  $\mathcal{H}$  that commutes with the representation  $\rho$ , with its opposite  $J\rho J^{-1}$ , with the chirality  $\chi$ , and with the Dirac operator  $\mathcal{D}$ . It follows that  $z$  involves  $2(1+N) = 8$  strictly positive numbers  $x, y_1, y_2, y_N, \tilde{x}, \tilde{y}_1, \tilde{y}_2, \tilde{y}_N$ ,

$$\begin{aligned} z &:= \begin{pmatrix} z_w & 0 \\ 0 & z_s \end{pmatrix}, \\ z_w &:= \begin{pmatrix} x/3 1_2 \otimes 1_N \otimes 1_3 & 0 & 0 & 0 \\ 0 & 1_2 \otimes y & 0 & 0 \\ 0 & 0 & x/3 1_2 \otimes 1_N \otimes 1_3 & 0 \\ 0 & 0 & 0 & y \end{pmatrix}, \\ z_s &:= \begin{pmatrix} \tilde{x}/3 1_2 \otimes 1_N \otimes 1_3 & 0 & 0 & 0 \\ 0 & 1_2 \otimes \tilde{y} & 0 & 0 \\ 0 & 0 & \tilde{x}/3 1_2 \otimes 1_N \otimes 1_3 & 0 \\ 0 & 0 & 0 & \tilde{y} \end{pmatrix}, \\ y &:= \begin{pmatrix} y_1 & 0 & 0 \\ 0 & y_2 & 0 \\ 0 & 0 & y_N \end{pmatrix}, \quad \tilde{y} := \begin{pmatrix} \tilde{y}_1 & 0 & 0 \\ 0 & \tilde{y}_2 & 0 \\ 0 & 0 & \tilde{y}_N \end{pmatrix}. \end{aligned}$$

The interpretation of these numbers is straightforward. The three  $y_j$  poise the weak interactions with the three lepton generations. The  $y_j$  enter independently because the Higgs scalar couples differently to the three leptons and in noncommutative geometry the Higgs is part of the

gauge interactions. The three  $\tilde{y}_j$  poise the ‘strong’ interactions with the three lepton generations. They do not drop out because of the  $b$  in  $\rho_s$ . However, they will only enter as sum: strong interactions are unbroken and do not generate a Higgs. We will denote  $\tilde{y} := \tilde{y}_1 + \tilde{y}_2 + \tilde{y}_N$  and there should be no risk of confusion.  $x$  and  $\tilde{x}$  poise weak and strong interactions with quarks. There is only one number per interaction because of the Cabibbo-Kobayashi-Maskawa mixing that we suppose non-degenerate.

### 3.2.2 The choice of a scalar product

We recall the internal scalar product  $\langle \omega, \kappa \rangle_z = \text{Re tr} [z\omega^* \kappa]$ ,  $\omega, \kappa \in \pi(\Omega\mathcal{A})$ . At this point comes the new lesson from noncommutative relativity. It tells us that we have forgotten an entire cone of other scalar products,

$$\langle \omega, \kappa \rangle_{z'} := \text{Re tr} [z'(\omega + J\omega J^{-1})^*(\kappa + J\kappa J^{-1})]$$

with additional  $1+N$  strictly positive constants  $x', y'_1, y'_2, y'_N$ ,

$$z' := \begin{pmatrix} z'_w & 0 \\ 0 & z'_s \end{pmatrix},$$

$$z'_w = Z'_s := \frac{1}{2}$$

$$\begin{pmatrix} x'/3 \mathbf{1}_2 \otimes \mathbf{1}_N \otimes \mathbf{1}_3 & 0 & 0 & 0 \\ 0 & \mathbf{1}_2 \otimes y' & 0 & 0 \\ 0 & 0 & x'/3 \mathbf{1}_2 \otimes \mathbf{1}_N \otimes \mathbf{1}_3 & 0 \\ 0 & 0 & 0 & y' \end{pmatrix}.$$

Indeed, in noncommutative relativity, the scalar product is not chosen, it is induced from the heat kernel calculation. The Dirac operator  $\mathcal{D}_{t,\text{cov}} = \mathcal{D}_t + \pi_t(A_t) + J_t \pi_t(A_t) J_t^{-1}$  leads to the scalar product with  $z'$ . We could obtain the one with  $z$  from another Dirac operator,  $\mathcal{D}_{t,\text{cov}} = \mathcal{D}_t + \pi_t(A_t)$ , but this latter is forbidden by the spectral principle: for a unitary  $u \in \mathcal{A}_t$ , the inner automorphism

$$\alpha_u : \rho_t(a) \in \rho_t(\mathcal{A}_t) \mapsto \rho_t(uau^*) \in \rho_t(\mathcal{A}_t)$$

induces a unitary operator  $U = \rho_t(u) J_t \rho_t(u) J_t^{-1}$  on  $\mathcal{H}_t$  satisfying

$$\begin{aligned} U \rho_t(a) U^* &= \alpha_u(\rho_t(a)), \quad \text{and} \\ U \mathcal{D}_t U^* &= \mathcal{D}_t + \pi_t(A_t) + J_t \pi_t(A_t) J_t^{-1}, \\ &\text{with } \pi_t(A_t) = \rho_t(u) [\mathcal{D}_t, \rho_t(u)^*], \end{aligned}$$

so  $\mathcal{D}_t$  and  $\mathcal{D}_{t,\text{cov}}$  have the same spectrum and the fluctuations of the metric are of the form  $\pi_t(A_t) + J_t \pi_t(A_t) J_t^{-1}$ .

Restricted to the Lie algebra  $\mathfrak{g}$ , we have a subtle nuance between the two invariant scalar products concerning the two  $u(1)$  factors. For  $z'$  in the center  $\mathbb{R}^+_{190}$ , we have  $\sin^2 \theta_w = 3/8$ , while for  $z$  in the center we will get  $\sin^2 \theta_w = 12/29$ . Note also that  $z'$  commutes with the real structure  $J$ , while  $z$  does not. If in doubt, stay out: we will use both cones simultaneously. Figure 2 is an artist’s view

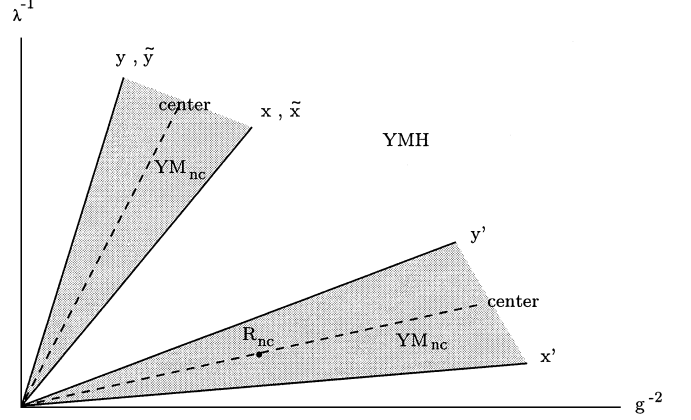


Fig. 2. The allowed cones of possible scalar products

on the role of the possible scalar products in noncommutative Yang-Mills theory:

$$\begin{aligned} \langle \omega, \kappa \rangle_{z,z'} &:= \text{Re tr} [z\omega^* \kappa] \\ &+ \text{Re tr} [z'(\omega + J\omega J^{-1})^*(\kappa + J\kappa J^{-1})], \\ \omega, \kappa &\in \pi(\Omega\mathcal{A}). \end{aligned} \quad (24)$$

### 3.2.3 The gauge couplings computation

We are ready to turn the crank. A long, but straight down the line computation leads to the physical couplings in terms of the fermionic mass matrix  $\mathcal{M}$  and the noncommutative couplings  $z, z'$ :

Here are a few purely algebraic intermediate steps:

$$\Omega_{\mathcal{D}}^1 \mathcal{A} = \left\{ i \begin{pmatrix} 0 & \rho_{wL}(h) \mathcal{M} & 0 & 0 \\ \mathcal{M}^* \rho_{wL}(\tilde{h}^*) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, h, \tilde{h} \in \mathbb{H} \right\}.$$

The Higgs being an anti-Hermitian 1-form

$$\begin{aligned} H &= i \begin{pmatrix} 0 & \rho_{wL}(h) \mathcal{M} & 0 & 0 \\ \mathcal{M}^* \rho_L(h^*) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\ h &= \begin{pmatrix} h_1 & -\bar{h}_2 \\ h_2 & -\bar{h}_1 \end{pmatrix} \in \mathbb{H} \end{aligned}$$

is parameterized by one complex doublet

$$\begin{pmatrix} h_1 \\ h_2 \end{pmatrix}, \quad h_1, h_2 \in \mathbb{C}.$$

The internal junk in degree two turns out to be

$$\mathcal{J}^2 = \left\{ i \begin{pmatrix} j \otimes \Delta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, j \in \mathbb{H} \right\}$$

$$\begin{aligned}
[N(x+x') + y + y']\alpha + & & y'\beta + & & Nx'\gamma = \frac{1}{2}(x+x') \\
2y'\alpha + [2N(x+x') + y + 3\tilde{y} + 6y']\beta + & & 2Nx'\gamma = & & (x+x') \\
Nx'\alpha + & & Nx'\beta + 2N(\tilde{x} + x')\gamma = & & x'.
\end{aligned} \tag{25}$$

with

$$\Delta := \frac{1}{2} \begin{pmatrix} (M_u M_u^* - M_d M_d^*) \otimes 1_3 & 0 \\ 0 & -M_e M_e^* \end{pmatrix}.$$

The homogeneous scalar variable is:

$$\Phi := H_i \mathcal{D} =: i \begin{pmatrix} 0 & \rho_{wL}(\phi) \mathcal{M} & 0 & 0 \\ \mathcal{M}^* \rho_{wL}(\phi^*) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\phi = \begin{pmatrix} \varphi_1 & -\bar{\varphi}_2 \\ \varphi_2 & \bar{\varphi}_1 \end{pmatrix} \in \mathbb{H},$$

and with  $\varphi := (\varphi_1, \varphi_2)^T$ , the internal field strength is:

$$\begin{aligned}
C := \delta H + H^2 = (1 - |\varphi|^2) & \begin{pmatrix} 1_2 \otimes \Sigma & 0 & 0 & 0 \\ 0 & \mathcal{M}^* \mathcal{M} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
\Sigma := \frac{1}{2} & \begin{pmatrix} (M_u M_u^* + M_d M_d^*) \otimes 1_3 & 0 \\ 0 & M_e M_e^* \end{pmatrix}.
\end{aligned}$$

Now, the chosen scalar product (24) appears in the long computation of  $\alpha C$  [19]. In the standard model,  $\alpha C$  has form

$$\alpha C = (1 - |\varphi|^2) m_t^2 \rho(\alpha 1_2, \beta, \gamma 1_3).$$

To compute the real numbers  $\alpha, \beta, \gamma$ , we neglect all fermion masses with respect to the top mass. This approximation is again good to  $m_b^2/m_t^2 = 0.0006$  [23]. In this approximation, the number of parameters reduces again, we are left with six parameters:  $x, \tilde{x}, y := y_1 + y_2 + y_N, x',$  and  $y' := y'_1 + y'_2 + y'_N, \tilde{y}$ , and  $\alpha, \beta, \gamma$  are determined by three linear equations (see (25) on top of the page).

The Higgs Lagrangian has the form:

$$\begin{aligned}
& \frac{2}{8\pi^2} (x+x') m_t^2 (D_\mu \varphi)^* D^\mu \varphi + \frac{1}{8\pi^2} k m_t^4 (1 - |\varphi|^2)^2 \\
& =: \frac{1}{2} (D_\mu \varphi_{\text{ph}})^* D^\mu \varphi_{\text{ph}} + \lambda |\varphi_{\text{ph}}|^4 - \frac{1}{2} \mu^2 |\varphi_{\text{ph}}|^2 \\
& \quad + \text{constant},
\end{aligned}$$

with

$$\begin{aligned}
k := & \frac{3}{2} (x+x') - 2Nx(\alpha^2 + \beta^2) - y(2\alpha^2 + \beta^2) - 4N\tilde{x}\gamma^2 \\
& - 3\tilde{y}\beta^2 - 2Nx'((\alpha + \gamma)^2 + (\beta + \gamma)^2) \\
& - y'(2(\alpha + \beta)^2 + 4\beta^2).
\end{aligned}$$

Therefore

$$\lambda^{-1} = \frac{2}{\pi^2} \frac{(x+x')^2}{k}, \tag{26}$$

$$\mu^2 = \frac{k}{x+x'} m_t^2. \tag{27}$$

Before computing the gauge couplings, we have to get rid of the unwanted  $u(1)$  in  $\mathfrak{g}$ . This is done by imposing the unimodularity condition,

$$\text{tr}[P(\rho(a, b, c) + J\rho(a, b, c)J^{-1})] = 0,$$

where  $P$  is the projection on  $\mathcal{H}_L \oplus \mathcal{H}_R$ , the space of particles. Note that this condition is equivalent to the condition of vanishing gauge anomalies [22]. Normalizing properly the gauge fields, we compute their couplings:

$$g_3^{-2} = \frac{1}{6\pi^2} N(\tilde{x} + x'), \tag{28}$$

$$g_2^{-2} = \frac{1}{8\pi^2} [N(x+x') + y + y'], \tag{29}$$

$$\begin{aligned}
g_1^{-2} = \frac{1}{8\pi^2} & \left[ Nx + \frac{2}{9} N\tilde{x} + \frac{11}{9} Nx' \right. \\
& \left. + \frac{1}{2} y + \frac{3}{2} \tilde{y} + 3y' \right].
\end{aligned} \tag{30}$$

### 3.3 Results

As for noncommutative relativity, we interpret the five constraints (26–30) in terms of running quantities at the noncommutative scale  $\Lambda$ . Since the flow of  $\mu^2$  is renormalization scheme dependent, we trade the running top mass for its Yukawa coupling,  $m_t = g_t v$ ,  $m_W = \frac{1}{2} g_2 v$ ,  $m_H = 2\sqrt{2}\lambda v$ ,  $v = \frac{1}{2} \frac{\mu}{\sqrt{\lambda}}$ ,

$$g_3(\Lambda)^{-2} = \frac{1}{6\pi^2} N(\tilde{x} + x'), \tag{31}$$

$$g_2(\Lambda)^{-2} = \frac{1}{8\pi^2} [N(\tilde{x} + x') + y + y'], \tag{32}$$

$$\begin{aligned}
g_1(\Lambda)^{-2} = \frac{1}{8\pi^2} & \left[ Nx + \frac{2}{9} N\tilde{x} + \frac{11}{9} Nx' \right. \\
& \left. + \frac{1}{2} y + \frac{3}{2} \tilde{y} + 3y' \right],
\end{aligned} \tag{33}$$

$$\lambda(\Lambda)^{-1} = \frac{2}{\pi^2} \frac{(x+x')^2}{k}, \tag{34}$$

$$g_t(\Lambda)^{-2} = \frac{1}{2\pi^2} (x+x'). \tag{35}$$

These Yang-Mills constraints are to be compared to the soft Einstein-Hilbert constraints

$$g_3(\Lambda)^{-2} = \frac{1}{9\pi^2} f_4 N x', \tag{36}$$

$$g_2(\Lambda)^{-2} = \frac{1}{12\pi^2} f_4 (N x' + y'), \tag{37}$$

$$g_1(\Lambda)^{-2} = \frac{1}{12\pi^2} f_4 \left( \frac{11}{9} N x' + 3y' \right), \tag{38}$$

$$\lambda(\Lambda)^{-1} = \frac{1}{\pi^2} f_4 x'. \tag{39}$$



The noncommutative Yang-Mills action has four additional parameters,  $x, \tilde{x}, y, \tilde{y}$ , but one additional constraint, on the top mass.

These results can be detailed at different levels, playing with  $z$  and  $z'$ .

– The original Connes-Lott model [1,2] used  $z_w =: z_1, z_s =: z_2, z' = 0$  and  $\Lambda = m_Z$ , i.e. tree level. It worked with a bimodule and had *two* spurious  $U(1)$  factors. Consequently its linear system (25) is slightly different, put  $\tilde{y} = 0$ , and the Higgs mass comes out [24]:

$$m_H^2 = 3 \frac{(m_t/m_W)^4 + 2(m_t/m_W)^2 - 1}{(m_t/m_W)^2 + 3} m_W^2, \\ m_H = 278 \text{ GeV},$$

for  $m_t = 175 \text{ GeV}$ .

– With the real structure [3], we are inflicted with only one spurious  $U(1)$ . If we put  $z' = 0$ , then we can solve the system (25) even without the approximation of a dominating top mass:

$$\alpha = \frac{1}{2} \frac{x}{Nx + y}, \quad \beta = \frac{1}{2} \frac{x}{Nx + \frac{1}{2}y + \frac{3}{2}\tilde{y}}, \quad \gamma = 0,$$

and at tree level [23]:

$$m_H^2 = 3m_t^2 - \left( 1 + \frac{g_2^{-2}}{g_1^{-2} - \frac{1}{6}g_3^{-2}} \right) m_W^2, \\ m_H = 289 \text{ GeV}$$

for  $m_t = 175 \text{ GeV}$ . In this case, we also have precise results with all fermion masses and mixings. The  $\tau$  mass renders the Higgs mass fuzzy with a relative uncertainty of the order of  $m_\tau^2/m_t^2$ , that is some tens of MeV, as  $y_3$  ranges from 0 to its maximal value.

– Our general analysis including  $z, z'$  and  $\Lambda$  starts with the inequality,

$$g_2(\Lambda) < \frac{2}{\sqrt{N}} g_t(\Lambda) \tag{40}$$

coming from equations (32) and (35). Identifying the pole masses of the  $W$  and top with their running masses at  $m_Z$ , *this inequality sets an upper bound  $\Lambda_{\max}$  on the noncommutative scale* shown in Fig. 3. This bound is rather sensitive to variations in the gauge couplings. The noncommutative Einstein-Hilbert action needs a scale  $\Lambda$  of at least  $10^{10} \text{ GeV}$  forcing a high top mass or slightly different gauge couplings as suggested anyhow by the stiff action.

In the presence of  $z$ , the top mass is a free parameter and  $z'$  is just a perturbation rendering the Higgs mass fuzzy. This comes from the fact that  $x'$  and  $y'$  are bounded from above,

$$x'_{\max} = \min\{2\pi^2 g_3(\Lambda)^{-2}, 2\pi^2 g_t(\Lambda)^{-2}\} \\ y'_{\max} = \min\{8\pi^2 g_2(\Lambda)^{-2} - 6\pi^2 g_t(\Lambda)^{-2}, \frac{16\pi^2}{5} g_1(\Lambda)^{-2} \\ - \frac{8\pi^2}{5} g_2(\Lambda)^{-2} - \frac{8\pi^2}{15} g_3(\Lambda)^{-2} - \frac{6\pi^2}{5} g_t(\Lambda)^{-2}\}$$

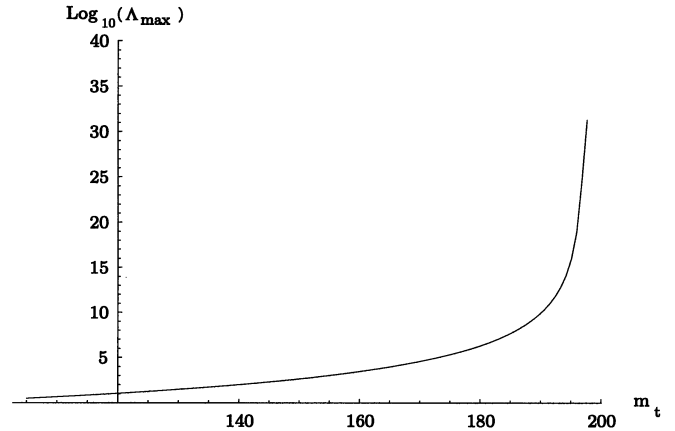


Fig. 3. The cutoff  $\Lambda_{\max}$  as function of the top mass

and that the Higgs mass decreases with  $x'$ , increases with  $y'$ . With  $m_t = 175 \pm 6 \text{ GeV}$ , this fuzziness is:

$$m_H = 289_{-5}^{+2} \text{ GeV for } \Lambda = m_Z, \\ m_H = 195_{-5}^{+0} \text{ GeV for } \Lambda = \Lambda_{\max} = 2 \cdot 10^5 \text{ GeV.} \tag{41}$$

This narrow interval accessible to the Higgs mass comes of course from a narrow interval accessible to the scalar selfcoupling,

$$\lambda(\Lambda) = (0.329 - 0.345)g_3^2 \quad \text{for } \Lambda = m_Z, \\ \lambda(\Lambda) = (0.313 - 0.317)g_3^2 \quad \text{for } \Lambda = \Lambda_{\max} = 2 \cdot 10^5 \text{ GeV.}$$

Finally, we have one other constraint in presence of  $z$  and  $z'$ ,

$$\frac{1}{15}g_3(\Lambda)^{-2} + \frac{1}{5}g_2(\Lambda)^{-2} + \frac{3}{20}g_t(\Lambda)^{-2} < \frac{2}{5}g_1(\Lambda)^{-2}.$$

We already had this same inequality [23] with  $z' = 0$ . For  $\Lambda = m_Z$  it means  $\sin^2 \theta_w < 0.54$  and remains harmless for higher  $\Lambda$ .

– To make contact with the noncommutative Einstein-Hilbert action, we put  $z = 0$ . Now the constraints on the gauge coupling are identical to those from the Einstein-Hilbert action and force upon us the big desert. In addition, the top mass is constrained,

$$g_t^2 = \frac{N}{3} g_3^2.$$

To compute the Higgs mass, we solve the system (25), which is simple, due to the approximation of a dominating top mass:

$$\alpha = 0, \quad \beta = 0, \quad \gamma = \frac{1}{2N}.$$

From (34) we have

$$\lambda(\Lambda) = \frac{3N - 2}{24} g_3^2 = \frac{7}{24} g_3^2. \tag{42}$$

This constraint is to be compared to the one from the Einstein-Hilbert action (39),  $\lambda(\Lambda) = \frac{N}{9} g_3^2$ . The two scalar

selfcouplings would coincide precisely if we had  $N = 6$  generations!

In terms of masses, we get in the soft case,  $\Lambda = 0.96 \cdot 10^{10}$  GeV:

$$m_t = 214 \pm 0 \pm 4 \text{ GeV}, \quad (43)$$

$$m_H = 227 \pm 0 \pm 4 \text{ GeV} \quad \text{soft YM}, \quad (44)$$

and in the stiff case,  $z' \in \rho(\text{center})$ :

$$m_t = 188 \pm 14 \pm 2 \pm 0 \text{ GeV}, \quad (45)$$

$$m_H = 198 \pm 8 \pm 2 \pm 0 \text{ GeV} \quad \text{stiff YM}. \quad (46)$$

For comparison, we recall the values from the Einstein-Hilbert action:

$$m_H = 190 \pm 0 \pm 1 \pm 4 \text{ GeV} \quad \text{soft EH}, \quad (47)$$

$$m_H = 182 \pm 10 \pm 2 \pm 7 \text{ GeV} \quad \text{stiff EH}. \quad (48)$$

The first error is from the uncertainty in the noncommutative scale,  $\Lambda = (10^{13}-10^{17})$  GeV, the second from the present experimental uncertainty in the gauge couplings,  $g_3 = 1.218 \pm 0.026$ , and the third is from the uncertainty in the top mass, if needed as input,  $m_t = 175 \pm 6$  GeV.

We give up the soft noncommutative Yang-Mills model since its top mass (43) must be bigger than 197 GeV, Fig. 3. It is too large and we retain only the stiff model. The stiff values for the top mass from Yang-Mills (45) seem in contradiction with the asymptotic value 197 GeV from Fig. 3. They are not, the asymptotic value is sensitive to changes in the gauge couplings which in the stiff case deviate slightly from the experimental values. The error of  $\pm 14$  GeV from the uncertainty in  $\Lambda$  tells us how close we are to the rim. Concerning the Higgs mass, the two allowed intervals, (48) from relativity and (46) from Yang-Mills have a non-empty intersection,

$$m_H = 188 - 201 \text{ GeV}.$$

This is the second pillar of the bridge we propose. All values of top and Higgs masses (43–48) are compatible with perturbation and stability in the energy range,  $m - Z < E < \Lambda$ , Fig. 6.

## 4 Conclusions

There is an old hand waving argument combining Heisenberg's uncertainty relation with the Schwarzschild radius which implies that the noncommutative scale  $\Lambda$  must be smaller than the Planck mass,  $10^{19}$  GeV. This is compatible with the numbers above and we have the following picture. Grand unification proposed the big desert, from the Higgs mass all the way up to  $\Lambda$  nothing new happens, no new particle, not even a break down of perturbation theory [25]. At  $\Lambda$ , a modest change of physics happens. The standard  $SU(3) \times SU(2) \times U(1)$  is unified into  $SU(5)$ . This adds a few more Yang-Mills and Higgs bosons to our boring world. These 'lepto-quarks' cause proton decay and make our world too exciting to be stable. Noncommutative geometry also has the big desert, but on its

other side a revolution in form of a truly noncommutative space-time of which the cheap tensor product between differential geometry and the internal space of the standard model is only a low energy mirage. We expect that crossing the scale  $\Lambda$  will induce 'noncommutative' threshold effects that are responsible for the small mismatch in the gauge couplings triangle in the  $E - g$  plane (Fig. 1) and for the small mismatch in the scalar selfcoupling and in the Yukawa coupling of the top. We also hope that the new geometry beyond  $\Lambda$  will solve our conceptual problems with quantum field theory, in particular in presence of gravity. But this is still troubled water.

The dream to connect general relativity and Yang-Mills theories is as old as Einstein. Elegant attempts have been proposed, Kaluza-Klein theories, Weyl's gravity, Poincaré gravity, Sakharov's induced gravity... Noncommutative geometry proposes another bridge, that stands so far thanks to a subtle conspiracy between the gauge couplings, the top mass and the number of generations. But it will fall soon if the Higgs mass does not cooperate.

## 5 Appendix

– **Gauge couplings:** The  $SU(2) \times U(1) \times SU(3)$  gauge couplings  $g_3, g_1, g_2$  are normalized by the scalar product

$$\langle \rho(a, b, c), \rho(a, b, c) \rangle_{z, z'} = \frac{1}{2} g_1^{-2} b \bar{b} + g_2^{-2} \text{tr}(a^* a) + g_3^{-2} \text{tr}(c^* c),$$

for  $(a, b, c) \in su(2) \oplus u(1) \oplus su(3)$ .

– **Higgs field:** The kinetic term of the scalar field  $\phi$  is normalized to  $\frac{1}{2}$  in the Lagrangian which is written as

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi^* \partial_\mu \phi + \lambda (\phi^* \phi)^2 - \frac{\mu^2}{2} (\phi^* \phi) + \dots$$

Moreover, if  $v$  is its expectation value, the relations between the gauge couplings  $g$ ., the self-coupling  $\lambda$  and the  $W$ , top, Higgs running masses are defined by

$$v = 2g_2^{-2} m_W,$$

$$\lambda = \frac{g_2^2 m_H^2}{32 m_W^2},$$

$$\mu = \frac{1}{\sqrt{2}} m_H,$$

$$g_t = v^{-1} m_t.$$

All these relations depend on the energy.

– **Renormalization procedure:** We adopt the mass independent  $\overline{MS}$  renormalization scheme in the approximation where all fermions masses are neglected but the top quark mass  $m_t$ . As running parameter associated to the energy  $E$ , we choose  $r = \log_{10} \left( \frac{E}{m_z} \right)$ . For the renormalization flow, the one-loop evolution equations of the above variables are the following first order differential equations

$$C g_1'(r) = \frac{41}{6} g_1(r)^3, \quad (49)$$

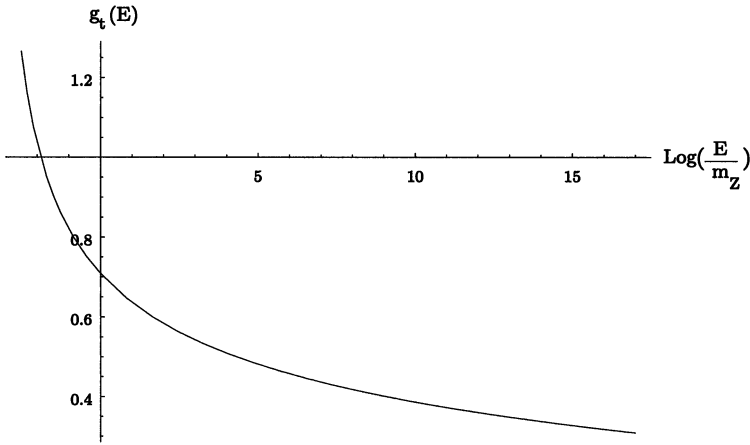


Fig. 4. The top coupling

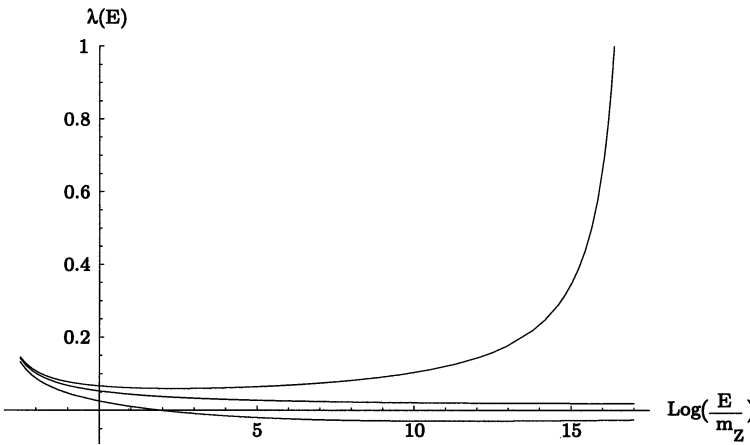


Fig. 5. The Higgs selfcoupling for  $m_H(m_Z) = 120$  (lower graph), 160 and 180 GeV (upper graph) for  $m_t(m_Z) = 175$  GeV

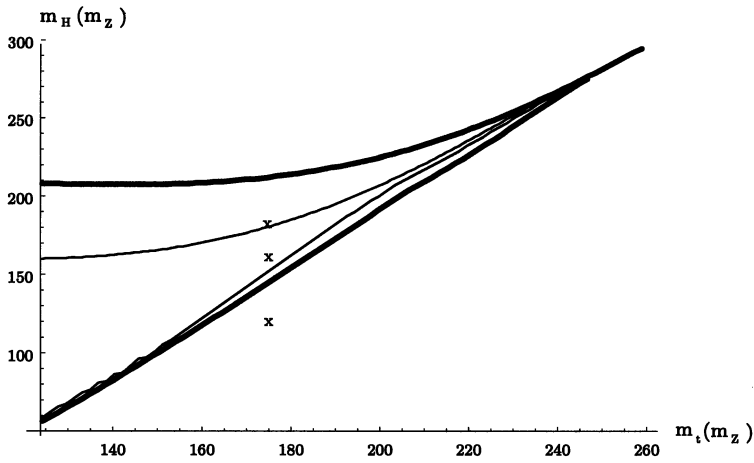


Fig. 6. Two allowed domains for the Higgs mass with  $\Lambda = 10^{10}$  GeV (thick lines) and  $\Lambda = 10^{19}$  GeV (thin lines): slices between the upper curves ( $\lambda < 1$ ) and the lower curves ( $\lambda > 0$ ), with points drawn at  $m_H(m_Z) = 120, 160, 180$  GeV, for  $m_t(m_Z) = 175$  GeV

$$C g_2'(r) = -\frac{19}{6} g_2(r)^3, \tag{50}$$

$$C g_3'(r) = -7g_3(r)^3, \tag{51}$$

$$C g_t'(r) = g_t(r) \left( -\frac{17}{12} g_1(r)^2 - \frac{9}{4} g_2(r)^2 - 8g_3(r)^2 + 9g_t(r)^2 \right) \tag{52}$$

$$C \lambda'(r) = \lambda(r) (-3g_1(r)^2 - 9g_2(r)^2 + 24g_t(r)^2 + 96\lambda(r))$$

$$+\frac{3}{32} g_1(r)^4 + \frac{9}{32} g_2(r)^4 - 6g_t(r)^4 + \frac{3}{16} g_1(r)^2 g_2(r)^2, \tag{53}$$

$$C \mu'(r) = \mu(r) \left( -\frac{3}{4} g_1(r)^2 - \frac{9}{4} g_2(r)^2 + 6g_t(r)^2 + 24\lambda(r) \right), \tag{54}$$

with  $C = \frac{16\pi^2}{\ln(19)}$ .

– **Initial conditions:** At  $r = 0$ , that is, at  $m_Z = 91.187$  GeV, we have [27]

$$g_1(0) = 0.3575 \pm 0.0001, \quad (55)$$

$$g_2(0) = 0.6507 \pm 0.0007, \quad (56)$$

$$g_3(0) = 1.218 \pm 0.0026, \quad (57)$$

$$m_t(0) = 175 \pm 6 \text{ GeV}, \quad (58)$$

$$m_W(0) = 80.33 \pm 0.15 \text{ GeV}. \quad (59)$$

So we get for the central values

$$v(0) = 246.903 \text{ GeV},$$

$$g_t(0) = 0.0040.$$

The last Eq. (54) decouples from the others: note that  $g_1, g_2, g_3, g_t, \lambda$  have no dimension while  $\mu$  is a mass. At this point, it is important to quote that in our renormalization scheme) quadratic divergences do not appear, only the logarithmic ones are retained. In order to avoid renormalization scheme ambiguities in the evolution of  $\mu^2$ , we neglect the threshold effects of the top and Higgs masses and we identify their pole masses  $m_p = m(m_p)$  with their running masses at the  $Z$  mass  $m(m_Z)$ . For these reasons, we will not use (54).

– **Figures:** All quantities not explicitly mentioned in a figure are put to their experimental values.

In Fig. 1, the intersections of the three coupling constants determine a triangle. Figure 4 shows the evolution of  $g_t$ . Clearly, the same evolution for the Higgs selfcoupling  $\lambda$  strongly depends in Fig. 5 of the Higgs initial value mass. This is due to the fact that the allowed domain for the Higgs mass in term of the top mass (Fig. 6) is the slice between the top curve which describes the perturbative (or triviality) condition ( $\lambda < 1$ ) and the down curve which describes the instability condition ( $\lambda > 0$ ) for energies between the  $Z$  mass and the Planck mass. Naturally, this slice depends on the choice of  $\Lambda$ . The three points in Fig. 6 are the initial conditions of the curves of Fig. 5. The upper point corresponds to the upper curve which is non perturbative and the lower point corresponds to the lower curve which is unstable. Since the experimental top mass (58) localizes the Higgs mass in a very narrow region, this figure is particularly significant if we believe in perturbation and stability throughout the big desert. It is important to note here that these two assumptions traditionally imposed by hand in a classical Yang-Mills-Higgs theory, are automatically satisfied in both noncommutative dreisätze: once the initial conditions (55–59) are admitted, the selfcoupling  $\lambda$  always stays in the stable and perturbative regime for energies between  $m_Z$  and  $\Lambda$ . Note for instance that, in the soft case,  $\Lambda = 0.96 \cdot 10^{10}$  GeV is small compared to the Planck mass. Figure 3 shows again that the noncommutative cutoff  $\Lambda_{\max}$  is very sensitive to a top mass around 197 GeV.

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